J Glob Optim (2008) 41:187–202
DOI 10.1007/s10898-007-9210-9

Sensitivity analysis of solution mappings of parametric vector quasi-equilibrium problems

Kenji Kimura · Jen-Chih Yao

Received: 16 December 2006 / Accepted: 28 June 2007 / Published online: 24 July 2007

Abstract In this article, we study the parametric vector quasi-equilibrium problem (PVQEP). We investigate existence of solution for PVQEP and continuities of the solution mappings of PVQEP. In particular, results concerning the lower semicontinuity of the solution mapping of PVQEP are presented.

Keywords Parametric vector quasi-equilibrium problem · Solution mapping · Upper semicontinuity · Lower semicontinuity

Mathematics Subject Classification (2000) 49J40 · 49K40 · 90C31

1 Introduction

Let X be nonempty subset of a real topological vector space and $\mathbb{Z}$ a real topological vector space. A set $\mathcal{C} \subset \mathbb{Z}$ is said to be a cone if $\lambda x \in \mathcal{C}$ for any $\lambda \geq 0$ and for any $x \in \mathcal{C}$. The cone $\mathcal{C}$ is called proper if it is not the whole space, i.e., $\mathcal{C} \neq \mathbb{Z}$. A cone $\mathcal{C}$ is said to be solid if it has nonempty interior, i.e., $\operatorname{int} \mathcal{C} \neq \emptyset$. Let $C : \mathcal{X} \to 2^{\mathbb{Z}}$, which has proper convex cone values. For any set $A \subset \mathbb{Z}$, we let $\operatorname{bd} A$ and $\operatorname{cl} A$ denote the boundary and closure of A, respectively. Also, we denote A^{c} the complement of the set A. For any set A of a real vector space, the convex hull of A, denoted by $\operatorname{co} A$, is the smallest convex set containing A. Furthermore, we denote zero vector of $\mathbb{Z}$ by $\theta_{\mathbb{Z}}$.

Let $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ and $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. For fixed $p \in \mathbb{P}$, the parametric vector quasi-equilibrium problem (PVQEP) is to find $x \in K(p, x)$ such that

$$\text{(PVQEP)} \quad f(p, x, y) \notin -\operatorname{int} C(p, x), \ \text{ for all } y \in K(p, x).$$

K. Kimura · J.-C. Yao (✉)
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan, R.O.C.
e-mail: yaojc@math.nsysu.edu.tw

K. Kimura
e-mail: kimura@math.nsysu.edu.tw

Let $S : \mathbb{P} \to 2^X$ be the set-valued mapping such that $S(p)$ is the solutions set of PVQEP for $p \in \mathbb{P}$, i.e.,

$$S(p) = \{x \in K(p, x) : f(p, x, y) \notin -\text{int}\, C(p, x), \text{ for all } y \in K\}.$$

In the literature, existence results for a (generalized) vector quasi equilibrium problems has been investigated. See, e.g., [1,2]. If for each fixed $p \in \mathbb{P}$, K and C have constant values for every $x \in X$, respectively, PVQEP reduces to a parametric vector equilibrium problem (PVEP). Existence of solution and closedness of solution mapping for PVEP has been studied in [3]. Continuity of solution mapping for PVEP has been studied in [4].

We observe that our results in this article can be employed to study the behavior of solution maps of parametric vector optimization, parametric vector variational inequalities, parametric vector equilibrium problems and those generalized problems and so on.

2 Preliminaries

Definition 2.1 (*C-continuity [5]*) Let X be a topological space and $\mathbb{Z}$ a topological vector space with a partial ordering defined by a proper solid convex cone C. Suppose that f is a vector-valued function from X to $\mathbb{Z}$. Then, f is said to be *C-continuous at $x \in X$*, if for any neighborhood $V_{f(x)} \subset \mathbb{Z}$ of $f(x)$, there exists a neighborhood $U_x \subset X$ of x such that $f(u) \in V_{f(x)} + C$ for all $u \in U_x$. Moreover a vector-valued function f is said to be *C-continuous in X* if f is C-continuous at every x on X.

Definition 2.2 (*Continuity for Set-valued mapping, See also [6]*) Let X and Y be two topological spaces, $T : X \to 2^Y$ a set-valued mapping.

(i) T is said to be *upper semicontinuous* (u.s.c. for short) at $x \in X$ if for each open set V containing $T(x)$, there is an open set U containing x such that for each $z \in U$, $T(z) \subset V$; T is said to be u.s.c. on X if it is u.s.c. at all $x \in X$.

(ii) T is said to be *lower semicontinuous* (l.s.c. for short) at $x \in X$ if for each open set V with $T(x) \cap V \neq \emptyset$, there is an open set U containing x such that for each $z \in U$, $T(z) \cap V \neq \emptyset$; T is said to be l.s.c. on X if it is l.s.c. at all $x \in X$.

(iii) T is said to be *continuous* at $x \in X$ if $T(x)$ is both u.s.c. and l.s.c.; T is said to be *continuous* on X if it is both u.s.c. and l.s.c. at each $x \in X$.

Proposition 2.1 *Let E be a nonempty subset of a topological space and $\mathbb{Z}$ a real topological vector space. Let $C : E \to 2^{\mathbb{Z}}$ with proper solid convex cone values and $W : E \to 2^{\mathbb{Z}}$ defined by $W(x) = \mathbb{Z} \setminus \big(-\text{int}\, C(x)\big)$. Suppose $f : E \to \mathbb{Z}$. If W is u.s.c. on E and for each $x \in E$, f is $(-C(x))$-continuous at x, then the set $\mathfrak{L} = \{x \in E : f(x) \in -\text{int}\, C(x)\}$ is open.*

Proof Let $x \in \mathfrak{L}$, and let $d \in \big(-\text{int}\, C(x)\big) \cap \big(f(x) + \text{int}\, C(x)\big)$. Then $d - \text{int}\, C(x)$ is a neighborhood of $f(x)$. Note that $d - \text{int}\, C(x) - \text{int}\, C(x) = -\text{int}\, C(x)$. Therefore because of $(-C(x))$-continuity of f at x, there exists a neighborhood U_1 of x such that

$$f(u) \in d - \text{int}\, C(x), \text{ for all } u \in U_1.$$

Note that $d - \text{int}\, C(x) \subset d - \text{cl}\, C(x)$ and that $\big(c - \text{cl}\, C(x)\big)^{\text{c}}$ is a neighborhood of $W(x)$. Since W is u.s.c. on E, there exists a neighborhood U_2 of x such that $d \in -\text{cl}\, C(x) \subset -\text{int}\, C(u)$ for all $u \in U_2$. Accordingly for each $u \in U_1 \cap U_2$ we have

$$f(u) \in -\text{int}\, C(u).$$

Hence $U_1 \cap U_2 \subset \mathfrak{L}$. Since $x \in \mathfrak{L}$ is arbitrary, $\mathfrak{L}$ is open. □

Definition 2.3 Let X and $\mathbb{Z}$ be two real vector spaces. Suppose that K is a nonempty convex set of X and that $T : X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$.

(i) T is said to be convex mapping on K if for each $x_1,\ x_2 \in K$ and $\mu \in [0, 1]$

$$T(\mu x_1 + (1-\mu)x_2) \supset \mu T(x_1) + (1-\mu)T(x_2);$$

(ii) T is said to be concave mapping on K if for each $x_1,\ x_2 \in K$ and $\mu \in [0, 1]$

$$T(\mu x_1 + (1-\mu)x_2) \subset \mu T(x_1) + (1-\mu)T(x_2);$$

(iii) T is said to be affine on K if T is a convex and concave mapping on K.

Definition 2.4 (*C-quasiconvexity, [7]*) Let X be a vector space and $\mathbb{Z}$ also a vector space with a partial ordering defined by a pointed convex cone C. Suppose that K is a convex subset of X and that f is a vector-valued function from K to $\mathbb{Z}$. Then, f is said to be *C-quasiconvex on K* if for each $z \in \mathbb{Z}$,

$$A(z) := \{x \in K : f(x) \in z - C\}$$

is convex or empty.

Definition 2.5 (*C-weak quasiconcavity*) Let X be a nonempty convex subset of a real topological vector space, $\mathbb{Z}$ a real topological vector space and $C : X \to 2^{\mathbb{Z}}$ with a proper solid convex cone values. Suppose $f : X \to \mathbb{Z}$. We say that f is *C-weakly quasiconcave* on X if for each $x_1, x_2 \in X$ such that $f(x_1) \notin -\text{int}\, C(x_1)$ and $f(x_2) \notin -\text{int}\, C(x_2)$, we have

$$f(x_\mu) \notin -\text{int}\, C(x_\mu), \text{ for all } x_\mu \in (x_1, x_2),$$

where $(x_1, x_2) = \{\alpha x_1 + (1-\alpha)x_2 : 0 < \alpha < 1\}$. We also say that f is *strictly C-weakly quasiconcave* on X if for each $x_1, x_2 \in X$ such that $f(x_1) \notin -\text{int}\, C(x_1)$ and $f(x_2) \notin -\text{int}\, C(x_2)$, we have

$$f(x_\mu) \notin -\text{cl}\, C(x_\mu), \text{ for all } x_\mu \in (x_1, x_2).$$

Remark 1 The concept of C-weak quasiconcavity in Definition 2.5 was originated from Ferro's properly C-quasiconcavity [7].

Example 1 Let $X = [0, 1] \times [0, 1]$ and $Z = \mathbb{R}^2$. Suppose that $f : X \to Z$ is defined by

$$f(x, y) = \begin{pmatrix} 2x - 1 \\ 2y - 1 \end{pmatrix}$$

and that $C : X \to 2^Z$ is defined by

$$C(x, y) = \left\{ (u, v) \in Z : \begin{pmatrix} x & y \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Then f is C-weakly quasiconcave on X.

Example 2 Let $X = [0, 1] \times [0, 1]$ and $Z = \mathbb{R}^2$. Suppose that $f : X \to Z$ is defined by

$$f(x, y) = \begin{pmatrix} 1 - x \\ 1 - y \end{pmatrix}$$

and that $C : X \to 2^Z$ is defined by

$$C(x, y) = \{(u, v) \in Z : ux + vy \geq 0\}$$

Then f is strictly C-weakly quasiconcave on X.

Definition 2.6 (*C-diagonally quasiconcavity*; see also *[8]*) Let X be a nonempty convex subset of a real vector space, $\mathbb{Z}$ a real vector space and $C : X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. We say that $f : X \times X \to \mathbb{Z}$ is *C-diagonally quasiconcave in its second argument*, if for any finite subset A of X and any $x \in \operatorname{co} A$, there exists $y \in A$ such that $f(x, y) \notin -\operatorname{int} C(x)$.

Proposition 2.2 *Let X be a nonempty convex subset of a real vector space, $\mathbb{Z}$ a real vector space and $C : X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Suppose that $f : X \times X \to \mathbb{Z}$. We also assume the following two conditions:*

(i) *for each $x \in X$ $f(x, x) \notin -\operatorname{int} C(x)$;*
(ii) *for each $x \in X$ $f(x, \cdot)$ is $C(x)$-quasiconvex on X.*

Then f is C-diagonally quasiconcave in its second argument.

Proof Suppose to the contrary that f is not C-diagonally quasiconcave in its second argument. Then there exist $x, x_1, \ldots, x_n \in X$ such that $x \in \operatorname{co}\{x_1, \ldots, x_n\}$ and $f(x, x_i) \in -\operatorname{int} C(x)$, for each $i = 1, \ldots, n$. Therefore by condition (ii), we have $f(x, x) \in -\operatorname{int} C(x)$. However this contradicts to condition (i). Accordingly f is C-diagonally quasiconcave in its second argument. □

Definition 2.7 (*Intersectional mapping [9]*) Let X be a topological space and $\mathbb{Z}$ a nonempty set. Let $T, G : X \to 2^{\mathbb{Z}} \setminus \{\emptyset\}$, respectively. We say G is an *intersectional mapping* of T, if for each $x \in X$ there exist a neighborhood $\mathcal{U}_x$ of x such that

$$G(x) \subset \bigcap_{u \in \mathcal{U}_x} T(u).$$

Proposition 2.3 [9, Proposition 2.2] *Let X be a nonempty subset of a real topological vector space and $\mathbb{Z}$ a real topological vector spaces, respectively. Let $C : X \to 2^{\mathbb{Z}}$, which has proper solid convex cone values. Suppose that $W : X \to 2^{\mathbb{Z}}$ defined by*

$$W(x) = \mathbb{Z} \setminus \operatorname{int} C(x)$$

has closed graph. Then C has at least one intersectional mapping, which has solid convex cone values.

Proposition 2.4 *Let E be a nonempty subset of a topological space and $\mathbb{Z}$ a real topological vector space. Let $C : E \to 2^{\mathbb{Z}}$ with proper solid convex cone values and $D : E \to 2^{\mathbb{Z}}$ an intersectional mapping of C with solid convex cone values. Suppose $f : E \to \mathbb{Z}$ and $W : E \to 2^{\mathbb{Z}}$, defined by $W(x) = \mathbb{Z} \setminus \big(-\operatorname{int} C(x)\big)$. If W has closed graph and for each $x \in E$ f is $(-D(x))$-continuous at x, then the set $\mathfrak{L} = \{x \in E : f(x) \in -\operatorname{int} C(x)\}$ is open.*

Proof Let $x \in \mathfrak{L}$ and let $d \in \big(-\text{int}\, C(x)\big) \cap \big(f(x) + \text{int}\, D(x)\big)$. Then $d - \text{int}\, D(x)$ is a neighborhood of $f(x)$. Note that $d - \text{int}\, D(x) - \text{int}\, D(x) = -\text{int}\, D(x)$. Therefore because of $(-D(x))$-continuity of f at x, there exists a neighborhood U_1 of x such that

$$f(u) \in d - \text{int}\, D(x), \text{ for all } u \in U_1.$$

Since W has closed graph, there exists a neighborhood U_2 of x such that $d \in -\text{int}\, C(u)$ for all $u \in U_2$. Since D is an intersectional mapping of C, there exists a neighborhood U_3 of x such that $\text{int}\, D(x) \subset \text{int}\, C(u)$ for all $u \in U_3$. Accordingly for each $u \in \bigcap_{i=1}^{3} U_i$ we have

$$f(u) \in -\text{int}\, C(u).$$

Hence $\bigcap_{i=1}^{3} U_i \subset \mathfrak{L}$. Since $x \in \mathfrak{L}$ is arbitrary, $\mathfrak{L}$ is open. □

Definition 2.8 Let X be a topological space and Y an nonempty set. A set-valued map $F : X \to 2^Y$ is said to *have open lower sections*, if the set $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open in X for every $y \in Y$.

Lemma 2.1 [10] *Let X be a topological space and Y a convex set of a real topological vector space. Let $F, G : X \to 2^Y$ be two set-valued maps with open lower sections. Then:*

(i) *the set-valued map $H : X \to 2^Y$, defined by $H(x) = \text{co}\,\big(F(x)\big)$ for all $x \in X$, has open lower sections;*
(ii) *the set-valued map $J : X \to 2^Y$, defined by $J(x) = F(x) \cap G(x)$ for all $x \in X$, has open lower sections.*

Lemma 2.2 [11, Fan-Browder fixed-point theorem] *Let X be a nonempty compact convex subset of a real Hausdorff topological vector space. Suppose that $F : X \to 2^X$ is a set-valued map with nonempty convex values and open lower sections. Then F has a fixed point.*

3 Existence of solution for PVQEP

In this section we drive some existence results for PVQEP.

Theorem 3.1 *Let X be a nonempty subset of a real Hausdorff topological vector space and $\mathbb{Z}$ a real topological vector space, $\mathbb{P}$ an index set and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ is a vector-valued function. Also we assume that the following conditions:*

(i) *for each $p \in \mathbb{P}$, $f(p, \cdot, \cdot)$ is C-diagonally quasiconcave in the third argument;*
(ii) *X is compact and convex;*
(iii) *$K(p, \cdot)$ has convex values and has open lower sections for each $p \in \mathbb{P}$;*
(iv) *for each fixed $p \in \mathbb{P}$ and $y \in X$, the set*

$$\{x \in X : f(p, x, y) \in -\text{int}\, C(p, x)\}$$

is open.

Then $S(p)$ is nonempty for each $p \in \mathbb{P}$.

Proof Let $p \in \mathbb{P}$ and for $x \in X$,

$$F(p, x) := \{y \in X : f(p, x, y) \in -\text{int}\, C(p, x)\}.$$

Then by condition (i), we have

$$x \notin \operatorname{co}\big(F(p,x)\big), \ \text{ for each } x \in X. \tag{1}$$

Let $G(p,x) = K(p,x) \cap \operatorname{co}\big(F(p,x)\big)$. By condition (iii), K has convex values and K has open lower sections. Hence there exists $x' \in X$ such that $x' \in K(p,x')$ by Lemma 2.2. If for every $x \in X$, $G(p,x) = \emptyset$, then $x' \in S(p)$. Thus we may suppose $G(p,x) \neq \emptyset$ for some $x \in X$. By condition (iv), F has open lower sections on X for each $p \in \mathbb{P}$. Therefore by Lemma 2.1, G has open lower sections. Clearly G has convex values. Let $H : X \to 2^X$ be defined by

$$H(p,x) = \begin{cases} G(p,x), & \text{if } G(p,x) \neq \emptyset, \\ K(p,x), & \text{otherwise.} \end{cases}$$

Hence H has convex values and open lower sections. Accordingly by Lemma 2.2, there exists $\hat{x} \in X$ such that $\hat{x} \in H(\hat{x})$. Because of (1), $\hat{x} \in K(p,\hat{x})$ and $G(p,\hat{x}) = \emptyset$. Hence $\hat{x} \in S(p)$. Therefore $S(p) \neq \emptyset$. Since $p \in \mathbb{P}$ is arbitrary, we have $S(p) \neq \emptyset$ for each $p \in \mathbb{P}$. □

Remark 2 By Proposition 2.2, condition (i) of Theorem 3.1 can be replaced by the following two conditions:

(C1) for each $p \in \mathbb{P}$ and $x \in X$, $f(p,x,x) \notin -\operatorname{int} C(p,x)$;
(C2) for each $p \in \mathbb{P}$ and $x \in X$, $f(p,x,\cdot)$ is $C(p,x)$-quasiconvex on X.

Also by Proposition 2.1, condition (iv) of Theorem 3.1 can be replaced by the following condition:

(C3) for each $p \in \mathbb{P}$ and $y \in X$, $f(p,\bullet,y)$ is $(-C(p,x))$-continuous on X.

Example 3 Let $\mathbb{P} = \{0\}$, $X = [0,1] \times [0,1]$ and $Z = \mathbb{R}^2$. Suppose that $K : \mathbb{P} \times X \to 2^X$, $C : \mathbb{P} \times X \to 2^Z$ and $f : \mathbb{P} \times X \times X \to Z$ are defined by

$$K(p,(x_1,x_2)) = \begin{cases} \left\{(x_1,x_2) \in X : x_2 \geq \frac{1}{2}\right\}, & x_1 < \frac{1}{2}, \\ \left\{(x_1,x_2) \in X : x_2 = \frac{1}{2}\right\}, & x_1 = \frac{1}{2}, \\ \left\{(x_1,x_2) \in X : x_2 \leq \frac{1}{2}\right\}, & x_1 > \frac{1}{2}, \end{cases}$$

$$C(p,x_1,x_2) = \begin{cases} \{(u,v) \in Z : u+v \leq 0\}, & \text{if } (x_1,x_2) = (0,0), \\ \{(u,v) \in Z : ux_1 + vx_2 \geq 0\}, & \text{otherwise.} \end{cases}$$

and

$$f(p,(x_1,x_2),(y_1,y_2)) = (2x_1 - y_1, 2x_2 - y_2),$$

respectively. Then by Theorem 3.1 and Remark 2, $S(p) \neq \emptyset$ for all $p \in \mathbb{P}$. Indeed,

$$\begin{aligned} S(p) &= \left\{(u,v) \in X : \left(u - \frac{1}{4}\right)^2 + \left(v - \frac{1}{4}\right)^2 \geq \frac{1}{8}\right\} \\ &\bigcap \left(\left\{(u,v) \in X : u > \frac{1}{2},\ v \geq \frac{1}{2}\right\} \cup \left\{(u,v) \in X : u < \frac{1}{2},\ v > \frac{1}{2}\right\}\right). \end{aligned}$$

To investigate upper and lower semicontinuity of S, we need to require closedness of $K(p, x)$ for each $p \in \mathbb{P}$ and $x \in X$. The following theorem is useful.

Theorem 3.2 *Let X be a nonempty subset of a real topological vector space and $\mathbb{Z}$ a real topological vector space, $\mathbb{P}$ an index set and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ is a vector-valued function. Also we assume that the following conditions:*

(i) *$S(p)$ is nonempty for each $p \in \mathbb{P}$;*
(ii) *the set $\{y \in X : f(p, x, y) \in -\mathrm{int}\, C(p, x)\}$ is open for each fixed $p \in \mathbb{P}$ and $x \in X$.*

Then

$$\overline{S}(p) = \{x \in \mathrm{cl}\, K(p, x) : f(p, x, y) \notin -\mathrm{int}\, C(p, x),\ \textit{for all } y \in \mathrm{cl}\, K(p, x)\}$$

is nonempty for each $p \in \mathbb{P}$.

Proof Let $p \in \mathbb{P}$ and $x \in S(p)$. Suppose to the contrary that there exists $y \in \mathrm{cl}\, K(p, x)$ such that $f(p, x, y) \in -\mathrm{int}\, C(p, x)$. Hence by condition (ii), there exists a neighborhood $\mathcal{U}$ of y such that

$$f(p, x, y') \in -\mathrm{int}\, C(p, x),\ \text{ for all } v' \in \mathcal{U}.$$

Clearly $\mathcal{U} \cap K(p, x) \neq \emptyset$. This contradicts to the fact that $x \in S(p)$. Hence $\overline{S}(p)$ is nonempty. Since $p \in \mathbb{P}$ is arbitrary, $\overline{S}(p)$ is nonempty for each $p \in \mathbb{P}$. □

The following result is a consequence of Theorems 3.1 and 3.2.

Theorem 3.3 *Let X be a nonempty subset of a real Hausdorff topological vector space and $\mathbb{Z}$ a real topological vector space, $\mathbb{P}$ an index set and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K, K' : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ is a vector-valued function. Also we assume conditions* (i)–(iii) *of Theorem* 3.1 *and the following conditions:*

(iv) *$K(p, x) = \mathrm{cl}\, K'(p, x)$ for each $p \in \mathbb{P}$ and $x \in X$;*
(v) *for each fixed $p \in \mathbb{P}$ and $y \in X$, the set $\{x \in X : f(p, x, y) \in -\mathrm{int}\, C(p, x)\}$ is open.*

Then $S(p)$ is nonempty for each $p \in \mathbb{P}$.

Remark 3 Conditions (iii) and (iv) and the condition that K has closed convex values and open lower sections are quite different. For example let $\mathbb{P} = \{1\},\ X = [0, 1],\ A = (0, \frac{1}{2})$ and $B = (\frac{1}{2}, 1)$. Suppose that $K : \mathbb{P} \times X \to 2^X$ is defined by

$$K(p, x) = \mathrm{cl}\left(xA + (1 - x)B\right).$$

Then K satisfies the former but not the latter.

Let $S'(p) := \{x \in K(p, x) : f(p, x, y) \notin -\mathrm{cl}\, C(p, x),\ \text{ for all } y \in K(p, x)\}$.

Theorem 3.4 *Let X be a nonempty subset of a real Hausdorff topological vector space and $\mathbb{Z}$ a real topological vector space, $\mathbb{P}$ an index set and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$ and $c : \mathbb{P} \times X \to \mathbb{Z}$ with $c(p, x) \in \mathrm{int}\, C(p, x)$ for each $p \in \mathbb{P}$ and $x \in X$. Suppose that $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ is a vector-valued function. Also we assume that the following conditions:*

(i) *for each* $p \in \mathbb{P}$, $f(p, \cdot, \cdot) - c(p, \cdot)$ *is* C*-diagonally quasiconcave in the third argument;*
(ii) X *is compact and convex;*
(iii) $K(p, \cdot)$ *has convex values and open lower sections for each* $p \in \mathbb{P}$*;*
(iv) *the set* $\{x \in X : f(p, x, y) - c(p, x) \in -\mathrm{int}\, C(p, x)\}$ *is open for each fixed* $p \in \mathbb{P}$ *and* $y \in X$.

Then $S'(p)$ *is nonempty for each* $p \in \mathbb{P}$.

Proof Let $f'(p, x, y) = f(p, x, y) - c(p, x)$. Hence by Theorem 3.1, for each $p \in \mathbb{P}$ there exists $x \in X$ such that $x \in K(p, x)$ and

$$f'(p, x, y) \notin -\mathrm{int}\, C(p, x), \text{ for all } y \in K(p, x),$$

i.e.,

$$f(p, x, y) - c(p, x) \notin -\mathrm{int}\, C(p, x), \text{ for all } y \in K(p, x). \tag{2}$$

Since $c(p, x) \in \mathrm{int}\, C(p, x)$ for each $p \in \mathbb{P}$ and $x \in X$, (2) implies

$$f(p, x, y) \notin -\mathrm{cl}\, C(p, x), \text{ for all } y \in K(p, x).$$

Hence $S'(p) \neq \emptyset$ for each $p \in \mathbb{P}$. □

Theorem 3.5 *Let X be a nonempty subset of a real Hausdorff topological vector space and* $\mathbb{Z}$ *a real topological vector space,* $\mathbb{P}$ *an index set and* $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ *with proper solid convex cone values. Let* $K, K' : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$ *and* $c : \mathbb{P} \times X \to \mathbb{Z}$ *with* $c(p, x) \in \mathrm{int}\, C(p, x)$ *for each* $p \in \mathbb{P}$ *and* $x \in X$. *Suppose that* $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ *is a vector-valued function. Also we assume conditions* (i)-(ii) *of Theorem* 3.4 *and the following conditions:*

(iii) $K'(p, \cdot)$ *has convex values and open lower sections for each* $p \in \mathbb{P}$*;*
(iv) $K(p, x) = \mathrm{cl}\, K'(p, x)$ *for each* $p \in \mathbb{P}$ *and* $x \in X$*;*
(v) *the set* $\{x \in X : f(p, x, y) - c(p, x) \in -\mathrm{int}\, C(p, x)\}$ *is open for each fixed* $p \in \mathbb{P}$ *and* $y \in X$.

Then $S'(p)$ *is nonempty for each* $p \in \mathbb{P}$.

Proof Let $f'(p, x, y) = f(p, x, y) - c(p, x)$. Then by Theorem 3.1, for each $p \in \mathbb{P}$ there exists $x \in X$ such that $x \in K'(p, x)$ and

$$f'(p, x, y) \notin -\mathrm{int}\, C(p, x), \text{ for all } y \in K'(p, x),$$

i.e.,

$$f(p, x, y) - c(p, x) \notin -\mathrm{int}\, C(p, x), \text{ for all } y \in K'(p, x).$$

By condition 3.5, we have

$$f(p, x, y) - c(p, x) \notin -\mathrm{int}\, C(p, x), \text{ for all } y \in K(p, x). \tag{3}$$

Since $c(p, x) \in \mathrm{int}\, C(p, x)$, (3) implies

$$f(p, x, y) \notin -\mathrm{cl}\, C(p, x), \text{ for all } y \in K(p, x).$$

Hence $S'(p)$ is nonempty for each $p \in \mathbb{P}$. □

4 Upper semicontinuity of the solution mapping

In this section we show that the solution mapping S of PVQEP is upper semicontinuous on $\mathbb{P}$ under suitable assumptions.

Theorem 4.1 *Let X be a nonempty subset of a real topological vector space and $\mathbb{Z}$ a real topological vector space, $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values and $\mathbb{P}$ a topological space. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ is a vector-valued function. Also we assume that the following conditions:*

(i) *$S(p)$ is nonempty for each $p \in \mathbb{P}$;*
(ii) *X is compact;*
(iii) *$K(p, x)$ is compact for each $p \in \mathbb{P}$ and $x \in X$;*
(iv) *K is u.s.c. on $\mathbb{P} \times X$;*
(v) *the set $\{(p, x, y) \in \mathbb{P} \times X \times X : f(p, x, y) \in -\operatorname{int} C(p, x)\}$ is open.*

Then S is u.s.c. on $\mathbb{P}$.

Proof Since X is compact, it suffices to show that S has closed graph. Let $p \in \mathbb{P}$. Suppose $\{p_\mu\} \subset \mathbb{P}$ is a net with $p_\mu \to p$ and $\{x_\mu\} \subset X$ is a net with $x_\mu \in S(p_\mu)$ for all $\mu \in \mathcal{M}$. Then $x_\mu \in K(p_\mu, x_\mu)$. By condition (ii), there exists a subnet $\{x_\nu\} \subset \{x_\mu\}$ such that $x_\nu \to x \in X$. Hence we may assume, without loss of generality, $p_\mu \to p$ and $x_\mu \to x$ with $x_\mu \in S(p_\mu)$. Because of conditions (ii), (iii) and (iv), K has closed graph. Note that $x_\mu \in S(p_\mu)$ implies $x_\mu \in K(p_\mu, x_\mu)$. Therefore we have $x \in K(p, x)$. Suppose to the contrary that $x \notin S(p)$. Then there exists $y \in K(p, x)$ such that

$$f(p, x, y) \in -\operatorname{int} C(p, x).$$

Then by condition (v), there exists a neighborhood $\mathcal{U}$ of (p, x, y) such that

$$f(p', x', y') \in -\operatorname{int} C(p', x'), \text{ for all } (p', x', y') \in \mathcal{U}.$$

This contradicts to the fact that $p_\mu \to p,\ x_\mu \to x$ and $x_\mu \in S(p_\mu)$ for all $\mu \in \mathcal{M}$. Hence $x \in S(p)$. Therefore S is u.s.c. on $\mathbb{P}$. □

The following result is a consequence of Theorem 4.1 and Proposition 2.1.

Theorem 4.2 *Let X be a nonempty subset of a real topological vector space and $\mathbb{Z}$ a real topological vector space, $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values and $\mathbb{P}$ a topological space. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ is a vector-valued function. Also we assume conditions* (i)–(iv) *of Theorem* 4.1 *and the following conditions:*

(v) *$W : \mathbb{P} \times X \to 2^Z$ defined by $W(p, x) = \mathbb{Z} \setminus \big(-\operatorname{int} C(p, x)\big)$ is u.s.c. on $\mathbb{P} \times X$;*
(vi) *f is $\big(-C(p, x)\big)$-continuous at (p, x, y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$.*

Then S is u.s.c. on $\mathbb{P}$.

Example 4 Let $\mathbb{P} = [0, 1]$, $\mathbb{X} = \mathbb{R}$, $X = [0, \frac{\pi}{2}]$, $Z = \mathbb{R}^2$, $A = (0, \frac{\pi}{4})$ and $B = (\frac{\pi}{4}, \frac{\pi}{2})$. Let

$$K(p,x) = \mathrm{cl}\left(p\frac{2x}{\pi}A + \frac{2(1-x)}{\pi}B\right),$$

$$C(p,x) = \begin{cases} \{(u,v) \in Z : u \geq 0\}, & x \in [0, \frac{\pi}{6}) \text{ and } p \in [0, \frac{1}{2}), \\ \{(u,v) \in Z : u \geq 0 \text{ and } v \geq 0\}, & x \in [\frac{\pi}{6}, \frac{\pi}{3}] \text{ or } p \in [\frac{1}{2}, 1], \\ \{(u,v) \in Z : v \geq 0\}, & x \in (\frac{\pi}{3}, 1]) \text{ and } p \in [0, \frac{1}{2}). \end{cases}$$

Suppose that

$$f(p,x,y) = \begin{cases} (0,-1), & x \leq y, x \in [0, \frac{\pi}{6}) \text{ and } p \in [0, \frac{1}{2}), \\ (0,0), & x \leq y, x \in [\frac{\pi}{6}, \frac{\pi}{3}] \text{ or } p \in [\frac{1}{2}, 1], \\ (-1,0), & x \leq y, x \in (\frac{\pi}{3}, 1]) \text{ and } p \in [0, \frac{1}{2}), \\ (-p,-p), & \text{otherwise}. \end{cases}$$

Then from Theorem 3.3 and Remark *2* we see that for each $p \in \mathbb{P}$, $S(p) \neq \emptyset$. We also observe that every condition of Theorem 4.2 is satisfied. Accordingly S is u.s.c. on $\mathbb{P}$. Indeed,

$$S(p) = \begin{cases} \{\frac{\pi}{6}\}, & p \in [0, 1), \\ \mathrm{co}\left\{\frac{\pi}{6}, \frac{\pi}{3}\right\}, & p = 1. \end{cases}$$

The following result is a consequence of Theorem 4.1 and Proposition 2.4.

Theorem 4.3 *Let X be a nonempty subset of a real topological vector space $\mathbb{X}$ and $\mathbb{Z}$ a real topological vector space, $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values and $\mathbb{P}$ a topological space. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ is a vector-valued function and that $D : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ is an intersectional mapping of C with solid convex cone values. Also we assume conditions* (i)–(iv) *of Theorem* 4.1 *and the following conditions:*

(v) *$W : \mathbb{P} \times X \to 2^Z$ defined by $W(p,x) = \mathbb{Z} \setminus \big(-\mathrm{int}\, C(p,x)\big)$ has closed graph;*
(vi) *f is $\big(-D(p,x)\big)$-continuous at (p,x,y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$.*

Then S is u.s.c. on $\mathbb{P}$.

Example 5 Let $\mathbb{P}$, $\mathbb{X}$, X and K be the same as those in Example *4*. Let

$$C(p,x) = \left\{(u,v) \in Z : \left\langle \begin{pmatrix} \cos(x+p) \\ \sin(x+p) \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle \geq 0 \right\},$$

$$D(p,x) = \left\{(u,v) \in Z : \begin{pmatrix} \cos x' & \sin x' \\ \cos x'' & \sin x'' \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2_+ \right\},$$

where $x' = x + p - \frac{\pi}{32}$, $x'' = x + p + \frac{\pi}{32}$. Suppose

$$f(p,x,y) = \begin{cases} (1+(y-x))\big(\cos(x+p), \sin(x+p)\big), & x \leq y, \\ p\big(-\cos(x+p), -\sin(x+p)\big), & x > y. \end{cases}$$

Then from Theorem 3.3 and Remark 2 we see that for each $p \in \mathbb{P}$, $S(p) \neq \emptyset$. We also observe that every condition of Theorem 4.3 is satisfied. Accordingly S is u.s.c. on $\mathbb{P}$. Indeed

$$S(p) = \begin{cases} \{\frac{\pi}{6}\} & p \in (0,1], \\ [\frac{\pi}{6}, \frac{\pi}{4}] & p = 0. \end{cases}$$

5 Lower semicontinuity of the solution mapping

We next establish that the solution mapping S of PVQEP is lower semicontinuous on $\mathbb{P}$ under suitable assumptions.

Theorem 5.1 *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ is a vector-valued function. Also we assume that the following conditions:*

(i) *$S'(p)$ is nonempty for each $p \in \mathbb{P}$;*
(ii) *$K(p,x)$ is convex and compact for each $p \in \mathbb{P}$ and $x \in X$;*
(iii) *for each $p \in \mathbb{P}$, $K(p,\cdot)$ is concave on X;*
(iv) *K is u.s.c. on $\mathbb{P} \times X$;*
(v) *$\mathcal{F} : \mathbb{P} \to 2^X$, defined by $\mathcal{F}(p) = \{x \in X : x \in K(p,x)\}$, is l.s.c. on $\mathbb{P}$ and $\mathcal{F}(p)$ is convex for each $p \in \mathbb{P}$;*
(vi) *the set $\{(p,x,y) \in \mathbb{P} \times X \times X : f(p,x,y) \notin -\mathrm{cl}\, C(p,x)\}$ is open;*
(vii) *f is C-weakly quasiconcave on $\mathbb{P} \times X \times X$.*

Then S is l.s.c. on $\mathbb{P}$.

Proof Suppose $p \in \mathbb{P}$. Let $\mathcal{V}$ be an open set of X with $\mathcal{V} \cap S(p) \neq \emptyset$ and $x_1 \in \mathcal{V} \cap S(p)$. If $x_1 \notin S'(p)$, we can choose $x_2 \in S'(p)$. Let $x_\mu = \mu x_1 + (1-\mu)x_2$, $\mu \in (0,1)$. Then by condition (vii) for each $\mu \in (0,1)$, we have

$$f(p, x_\mu, y) \notin -\mathrm{cl}\, C(p, x_\mu), \text{ for all } y \in \mu K(p,x_1) + (1-\mu)K(p,x_2).$$

Since $\mathcal{F}(p)$ is convex for each $p \in \mathbb{P}$, $x_\mu \in K(p, x_\mu)$. Also since $K(p,\cdot)$ is concave for each $p \in \mathbb{P}$,

$$K(p, x_\mu) \subset \mu K(p,x_1) + (1-\mu)K(p,x_2).$$

Hence we have $x_\mu \in S'(p)$ for all $\mu \in (0,1)$. Thus there exists $x \in \mathcal{V} \cap S'(p) \cap \mathrm{co}\,\{x_1, x_2\}$. If $x_1 \in S'(p)$, then let $x = x_1$.

By condition (vi) for each $y \in K(p,x)$ there exist corresponding neighborhoods P_y of p, U_y of u, and V_y of y such that

$$f(q,u,v) \notin -\mathrm{cl}\, C(q,u), \text{ for all } q \in P_y, \ u \in U_y \text{ and } v \in V_y.$$

Since $K(p,x)$ is compact, there exist $\{y_1,\ldots,y_n\}\subset K(p,x)$ such that $\bigcup_{i=1}^n V_{y_i}\supset K(p,x)$. Therefore we have

$$f(q,u,v)\notin -\mathrm{cl}\,C(q,u),\ \text{for all } q\in\bigcap_{i=1}^n P_{y_i},\ u\in\bigcap_{i=1}^n U_{y_i} \text{ and } v\in\bigcup_{i=1}^n V_{y_i}.$$

By condition (iv) there exist neighborhoods P of p and U of x such that

$$K(q,u)\subset\bigcup_{i=1}^n V_{y_i},\ \text{for all } q\in P \text{ and } u\in U.$$

Because of condition (v) there exists a neighborhood P' of p such that for each $q\in P'$ we have

$$u'\in K(q,u'),\ \text{for some } u'\in\bigcap_{i=1}^n U_{y_i}\cap\mathcal{V}.$$

Let $\mathcal{P}=\bigcap_{i=1}^n P_{y_i}\cap P\cap P'$. For each $p'\in\mathcal{P}$, there exists corresponding $x'\in\bigcap_{i=1}^n U_{y_i}\cap\mathcal{V}$ such that $x'\in K(p',x')$ and

$$f(p',x',y')\notin -\mathrm{cl}\,C(p',x'),\ \text{for all } y'\in K(p',x').$$

Therefore for each $p'\in\mathcal{P},\ S(p')\cap\mathcal{V}\neq\emptyset$. Thus S is l.s.c. at p. Since $p\in\mathbb{P}$ is arbitrary, S is l.s.c. on $\mathbb{P}$. □

Next we investigate condition (vi).

Proposition 5.1 *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C:\mathbb{P}\times X\to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K:\mathbb{P}\times X\to 2^X\setminus\{\emptyset\}$. Suppose that $f:\mathbb{P}\times X\times X\to\mathbb{Z}$ is a vector-valued function. Also we assume that the following conditions:*

(i) *f is $C(p,x)$-continuous at (p,x,y) for every $p\in\mathbb{P}$, $x\in X$ and $y\in X$;*
(ii) *C is u.s.c. on $\mathbb{P}\times X$.*

Then the set $\mathfrak{U}=\{(p,x,y)\in\mathbb{P}\times X\times X:f(p,x,y)\notin -\mathrm{cl}\,C(p,x)\}$ is open.

Proof Let $x\in\mathfrak{U}$, i.e., $f(p,x,y)\notin -\mathrm{cl}\,C(p,x)$ and let

$$d\in\big((-\mathrm{cl}\,C(p,x))^{\mathrm{c}}\big)\cap\big(f(p,x,y)-\mathrm{int}\,C(p,x)\big).$$

Then $d+\mathrm{int}\,C(p,x)$ is a neighborhood of $f(p,x,y)$. Note that

$$d+\mathrm{int}\,C(p,x)+\mathrm{int}\,C(p,x)=d+\mathrm{int}\,C(p,x).$$

Therefore because of $C(p,x)$-continuity of f at (p,x,y), there exists a neighborhood $\mathcal{U}$ of (p,x,y) such that

$$f(q,u,v)\in d+\mathrm{int}\,C(p,x),\ \text{for all } (q,u,v)\in\mathcal{U}.$$

Note that $d+\mathrm{int}\,C(p,x)\subset d+\mathrm{cl}\,C(p,x)$ and that $\big(d+\mathrm{cl}\,C(p,x)\big)^{\mathrm{c}}$ is a neighborhood of $C(p,x)$. Since C is u.s.c. on $\mathbb{P}\times X$, there exists a neighborhood $\mathcal{V}$ of (p,x) such that

$$\big(d+\mathrm{cl}\,C(p,x)\big)\cap\big(-\mathrm{cl}\,C(q,u)\big)=\emptyset,\ \text{for all } (q,u)\in\mathcal{V}.$$

Accordingly for each $(q,u,v)\in\mathcal{U}\cap\big(\mathcal{V}\times X\big)$ we have

$$f(q,u,v)\notin -\mathrm{cl}\,C(q,u).$$

Hence $\mathcal{U}\cap\big(\mathcal{V}\times X\big)\subset\mathfrak{U}$. Since $(p,x,y)\in\mathfrak{U}$ is arbitrary, $\mathfrak{U}$ is open. □

Proposition 5.2 *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ is a vector-valued function and that $D : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ is a intersectional mapping of C with solid convex cone values. Also we assume that the following conditions:*

(i) *f is $D(p, x)$-continuous at (p, x, y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*
(ii) *C has closed graph.*

Then the set $\mathfrak{U} = \{(p, x, y) \in \mathbb{P} \times X \times X : f(p, x, y) \notin -\mathrm{cl}\, C(p, x)\}$ is open.

Proof Let $(p, x, y) \in \mathfrak{U}$. Then we have

$$f(p, x, y) \notin -\mathrm{cl}\, C(p, x).$$

Let $d \in \big(-clC(p, x)\big)^{c} \cap \big(f(p, x, y) - \mathrm{int}\, D(p, x)\big)$. Since f is $D(p, x)$-continuous at (p, x, y), there exists a neighborhood $\mathcal{U}$ of (p, x, y) such that

$$f(q, u, v) \in d + \mathrm{int}\, D(p, x), \text{ for all } (q, u, v) \in \mathcal{U}.$$

Since C has closed graph and D is an intersectional mapping of C, there exists a neighborhood $\mathcal{V}$ of (p, x) such that

$$\big(d + \mathrm{cl}\, D(p, x)\big) \cap \big(-\mathrm{cl}\, C(q, u)\big) = \emptyset, \text{ for all } (q, u) \in \mathcal{V}.$$

Accordingly for each $(q, u, v) \in \mathcal{U} \cap \big(\mathcal{V} \times X\big)$ we have

$$f(q, u, v) \notin -\mathrm{cl}\, C(q, u), \text{ for all } (q, u, v) \in \mathcal{U} \cap \big(\mathcal{V} \times X\big).$$

Hence $\mathcal{U} \cap \big(\mathcal{V} \times X\big) \subset \mathfrak{U}$. Since $(p, x, y) \in \mathfrak{U}$ is arbitrary, $\mathfrak{U}$ is open. □

Theorem 5.2 *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ is a vector-valued function. Also we assume conditions* (i)–(v) *of Theorem* 5.1 *and the following conditions:*

(vi) *f is $C(p, x)$-continuous at (p, x, y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*
(vii) *C is u.s.c. on $\mathbb{P} \times X$;*
(viii) *f is C-weakly quasiconcave on $\mathbb{P} \times X \times X$.*

Then S is l.s.c. on $\mathbb{P}$.

Proof The results follows from Theorem 5.1 and Proposition 5.1. □

Theorem 5.3 *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ is a vector-valued function and that $D : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ is a intersectional mapping of C with solid convex cone values. Also we assume conditions (i)–(v) of Theorem 5.1 and the following conditions:*

(vi) *f is $D(p, x)$-continuous at (p, x, y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*
(vii) *C has closed graph;*
(viii) *f is C-weakly quasiconcave on $\mathbb{P} \times X \times X$.*

Then S is l.s.c. on $\mathbb{P}$.

Proof The results follows from Theorem 5.1 and Proposition 5.2. □

Example 6 Let $\mathbb{P},\ \mathbb{X},\ X,\ K,\ C$ and D be the same as those in Example 5. Suppose

$$f'(p,x,y) = (x-\frac{\pi}{4})\big(\sin(x+p), -\cos(x+p)\big)+(y-x)\big(\cos(x+p), \sin(x+p)\big)$$

and

$$f(p,x,y) = \begin{cases} f'(p,x,y) - p(p-2)\big(\cos(x+p), \sin(x+p)\big) & p < 1, \\ f'(p,x,y) & p = 1. \end{cases}$$

Then by Theorem 3.5 and Remark 2, we have

$$S'(p) = \{x \in K(p,x) : f(p,x,y) \notin -\mathrm{cl}\, C(p,x),\ \text{for all } y \in K(p,x)\} \neq \emptyset$$

for each $p \in \mathbb{P}$. Hence by Theorem 5.3, S is l.s.c. on $\mathbb{P}$. Indeed,

$$S(p) = \begin{cases} \big\{x \in X : \frac{\pi}{6} \le x \le \frac{\pi}{6} + \frac{2}{3}p(2-p)\big\}, & x \in [0,1), \\ \big\{x \in X : x = \frac{\pi}{6}\big\}, & x = 1. \end{cases}$$

6 Continuity of the solution mapping

By combining results established in Sects. 4 and 5, we have the following continuity results of the solution mapping of PVQEP.

Theorem 6.1 *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ is a vector-valued function. Also we assume that the following conditions:*

- (i) *$S'(p)$ is nonempty for each $p \in \mathbb{P}$;*
- (ii) *X is compact;*
- (iii) *$K(p,x)$ is convex and compact for each $p \in \mathbb{P}$ and $x \in X$;*
- (iv) *for each $p \in \mathbb{P}$, $K(p,\cdot)$ is concave on X;*
- (v) *K is u.s.c. on $\mathbb{P} \times X$;*
- (vi) *$\mathcal{F} : \mathbb{P} \to 2^X$, defined by $\mathcal{F}(p) = \{x \in X : x \in K(p,x)\}$, is l.s.c. on $\mathbb{P}$ and $\mathcal{F}(p)$ is convex for each $p \in \mathbb{P}$;*
- (vii) *the set $\{(p,x,y) \in \mathbb{P} \times X \times X : f(p,x,y) \in -\mathrm{int}\, C(p,x)\}$ is open;*
- (viii) *the set $\{(p,x,y) \in \mathbb{P} \times X \times X : f(p,x,y) \notin -\mathrm{cl}\, C(p,x)\}$ is open;*
- (ix) *f is C-weakly quasiconcave on $\mathbb{P} \times X \times X$.*

Then S is continuous on $\mathbb{P}$.

Proof The result follows from Theorems 4.1 and 5.1. □

Theorem 6.2 *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ is a vector-valued function. Also we assume that the following conditions:*

- (i) *$S(p)$ is nonempty for each $p \in \mathbb{P}$;*
- (ii) *X is compact;*

(iii) *$K(p,x)$ is convex and compact for each $p \in \mathbb{P}$ and $x \in X$;*
(iv) *for each $p \in \mathbb{P}$, $K(p,\cdot)$ is concave on X;*
(v) *K is u.s.c. on $\mathbb{P} \times X$;*
(vi) *$\mathcal{F} : \mathbb{P} \to 2^X$, defined by $\mathcal{F}(p) = \{x \in X : x \in K(p,x)\}$, is l.s.c. on $\mathbb{P}$ and $\mathcal{F}(p)$ is convex for each $p \in \mathbb{P}$;*
(vii) *the set $\{(p,x,y) \in \mathbb{P} \times X \times X : f(p,x,y) \in -\mathrm{int}\, C(p,x)\}$ is open;*
(viii) *the set $\{(p,x,y) \in \mathbb{P} \times X \times X : f(p,x,y) \notin -\mathrm{cl}\, C(p,x)\}$ is open;*
(ix) *f is strictly C-weakly quasiconcave on $\mathbb{P} \times X \times X$.*

Then S is continuous on $\mathbb{P}$.

Proof If $S(p)$ is singleton for some $p \in \mathbb{P}$, then S is u.s.c. at p implies that S is continuous at p. Hence we obtain the result from Theorems 4.1 and Theorem 5.1. □

Theorem 6.3 *Let X and $\mathbb{P}$ be two nonempty subsets of two real topological vector spaces, respectively, $\mathbb{Z}$ a real topological vector space and $C : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ with proper solid convex cone values. Let $K : \mathbb{P} \times X \to 2^X \setminus \{\emptyset\}$. Suppose that $f : \mathbb{P} \times X \times X \to \mathbb{Z}$ is a vector-valued function and that $D : \mathbb{P} \times X \to 2^{\mathbb{Z}}$ is a intersectional mapping of C with solid convex cone values. Also we assume conditions (i)-(v) of Theorem 6.1 and the following conditions:*

(vi) *$\mathcal{F} : \mathbb{P} \to 2^X$, defined by $\mathcal{F}(p) = \{x \in X : x \in K(p,x)\}$, is l.s.c. on $\mathbb{P}$ and $\mathcal{F}(p)$ is convex for each $p \in \mathbb{P}$;*
(vii) *f is $D(p,x)$-continuous at (p,x,y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*
(viii) *f is $\big(-D(p,x)\big)$-continuous at (p,x,y) for every $p \in \mathbb{P}$, $x \in X$ and $y \in X$;*
(ix) *C has closed graph;*
(x) *$W : \mathbb{P} \times X \to 2^Z$ defined by $W(p,x) = \mathbb{Z} \setminus \big(-\mathrm{int}\, C(p,x)\big)$ has closed graph;*
(xi) *f is C-weakly quasiconcave on $\mathbb{P} \times X \times X$.*

Then S is continuous on $\mathbb{P}$.

Proof The result follows from Theorems 4.3 and 5.3. □

We remark that results presented in Sects. 3–6 can be used to derive corresponding results for scalar parametric quasi-equilibrium problems.

Acknowledgement This research was partially supported by a grant from NSC. The authors also thank the referees for their helpful comments and suggestions which improved the original manuscript greatly.

References

1. Fang, Y.P., Huang, N.J.: Existence of solutions to generalized vector quasi-equilibrium problems with discontinuous mappings. Acta Math. Sin. (Engl.–Ser.) **22**(4), 1127–1132 (2006)
2. Peng, J.W., Zhu, D.L.: Generalized vector quasi-equilibrium problems with set-valued mappings. J. Inequal. Appl. **2006**, 1–12 (2006)
3. Kimura, K., Liou, Y.C., Yao, J.C.: A parametric equilibrium problem with applications to optimization problems under equilibrium constraints. Nonlinear Convex Anal. **7**(2), 237–243 (2006)
4. Kimura, K., Yao, J.C.: Sensitivity analysis of vector equilibrium problems. Taiwanese J. Math. (2008) (to appear)
5. Luc, D.T.: Theory of Vector Optimization: Lecture Notes in Economics and Mathematical Systems. Springer-Verlag Berlin Heidelberg (1989)
6. Berge, C.: Topological Space. Oliver&Boyd, Edinburgh and London (1963)
7. Ferro, F.: A minimax theorem for vector-valued functions. J. Optim. Theory Appl **60**(1), 19–31 (1989)
8. Hou, S.H., Yu, H., Chen, G.Y.: On vector-quasi-equilibrium problems with set-valued maps. J. Optim. Theory Appl. **119**(3), 485–498 (2003)

9. Kimura, K., Yao, J.C.: Semicontinuity of solution mappings of parametric generalized vector equilibrium problems. J. Optimiz. Theory. App. (to appear)
10. Tian, G.: Generalized quasi-variational-like inequality problem. Math. Oper. Res. **18**, 213–225 (1993)
11. Takahashi, W.: Nonlinear functional analysis—fixed point theory and its applications—. Yokohama Publishers, Yokohama (2000)